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Vibrations of rotating cross-ply laminated circular cylindrical shells with stringer and ring stiffeners

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Abstract

In this paper, the vibration analysis of simply supported rotating cross-ply laminated cylindrical shells with axial and circumferential stiffeners, i.e., stringers and rings, is presented using an energy approach. The effects of these stiffeners are evaluated via two methods, namely: by a variational formulation with individual stiffeners treated as discrete elements; and by an averaging method whereby the properties of the stiffeners are averaged over the shell surface. The effects of initial hoop tension, centrifugal and Coriolis forces due to the rotation are considered in the present formulation. Also, stiffener eccentricity is accounted for. The present formulation is verified by comparison with experimental and numerical results available in open literature. Excellent agreement is observed and a new range of results is presented for rotating shells which can be used as a benchmark to approximate solutions. Detailed studies on the depth-to-width ratios of the stiffeners on the forward and backward frequencies were conducted for the transverse, circumferential and longitudinal modes. © 2001 Published by Elsevier Science Ltd.

Keywords: Free vibration; Rotating composite laminated cylindrical shells; Stringer/ring stiffeners; Variational formulation; Energy method

1. Introduction

The free vibration of stiffened cylindrical shells has been investigated since the 1960s by a number of researchers. Mikulas and McElman (1965) investigated the free vibration of eccentrically stiffened simply supported cylindrical shells by averaging the stiffeners properties over the surface of the shells and found that the eccentricity could have significant effects on natural frequencies. Egle and Sewall (1968) extended this study by analyzing the free vibration of orthogonally stiffened cylindrical shells with stiffeners treated as discrete elements. The Rayleigh–Ritz procedure was employed with assumed displacement functions allowing for the coupling with both axial and circumferential modes. The effects of in-plane and rotary inertia on the natural frequencies of eccentrically stiffened shells were examined by Parthan and Johns (1970).

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Results were also presented by Rinehart and Wang (1972) for the vibration of simply supported cylindrical shells with longitudinal stiffeners. A theoretical-cum-experimental investigation into the vibration of axially loaded stiffened cylindrical shells was provided by Rosen and Singer (1974) using Donell and Flügge theories. Mustafa and Ali (1989) presented an energy method for free vibration analysis of stiffened circular cylindrical shells. The analysis took into account the flexure and extension of the shell and the flexure, extension and torsion of the stiffeners.

The behavior of rotating cylindrical shells was first investigated by Bryan (1890) and it was in this work that travelling modes were first elucidated. The effects of Coriolis and centrifugal forces in rotating shell structures have been examined by DiTaranto and Lessen (1964), Srinivasan and Lauterbach (1971) and Huang and Soedel (1988). Natural frequencies of rotating laminated cylindrical shells were obtained by Lam and Loy (1995) via four different thin shell theories. Huang and Chen (1996) used a modified receptance method for the vibration analysis of rotating cylindrical shells with internal, symmetric and external ring stiffeners. It was observed here that for the forward modes, the ring stiffeners stiffened only the modes where the circumferential wave number was greater than one. The backward modes were however stiffened by the attached rings for all circumferential wave numbers. Lee and Kim (1998, 1999) used the energy method to examine the effects of boundary conditions on the free vibration of rotating composite cylindrical shells with orthogonal stiffeners.

In the present study, an energy-based method is formulated to determine the effects of axial loading, lamination scheme and rotating speed on the frequency characteristics of stiffened rotating cross-ply laminated cylindrical shells. Both stringer and ring stiffener types are considered with stiffener eccentricity also taken into consideration. Two methods are presented. Firstly, a variational formulation with individual stiffeners treated as discrete elements, and secondly, an averaging method where the properties of the stiffeners are averaged over the shell surface. For rotating shells without stiffeners, numerical results for cross-ply laminated cylindrical shells without axial loading are compared with those given by Lam and Loy (1995). For non-rotating cases, present results for stiffened shells with external/internal stringers as well as orthogonal stiffeners are compared with those of Egle and Sewall (1968), Rinehart and Wang (1972), Mustafa and Ali (1989), and also those published in an ESDU (1982) report. Further, the effects of the depth-to-width ratios of the stiffeners on the forward and backward frequencies were investigated for the transverse, circumferential and longitudinal modes.

2. Theoretical formulation

The stiffened cylindrical shell, as shown in Fig. 1, is considered to be thin, laminated and composed of an arbitrary number layers with parameters length L, radius R, thickness h, and is rotating about the x-axis at constant angular velocity Ω. A coordinate system (x, θ, z) is fixed on the middle surface of the shell. The displacements of the shell in the x, θ and z directions are denoted by u, v, and w respectively. The depths of the stringer and ring are denoted by d_s and d_r, respectively, and the corresponding widths by b_s and b_r, respectively. The displacements from the middle surface of the shell to the centroid of the stringer and ring are denoted by z_s and z_r, respectively. The velocity and displacement vectors of a point on the shell are denoted by $\overline{V}$ and $\bar{r}$, respectively. The velocity at any point of the shell in vector form is given by

$$\overline{V} = \dot{\bar{r}}(\Omega = 0) + (\Omega\bar{i} \times \bar{r}) \tag{1}$$

in which the displacement vector $\bar{r}$ is written as

$$\bar{r} = u\bar{i} + v\bar{j} + w\bar{k} \tag{2}$$

where $\bar{i}$, $\bar{j}$ and $\bar{k}$ respectively denote the unit vectors in the x, θ and z directions when $\Omega = 0$. Substituting Eq. (2) into Eq. (1), the velocity vector can be expressed as

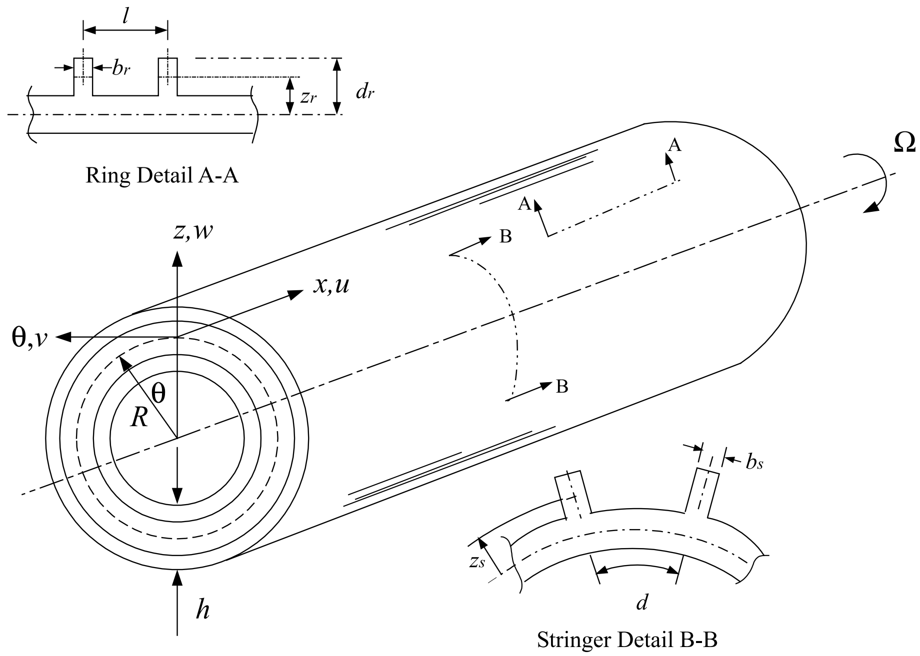

Fig. 1. Geometry of the stiffened rotating laminated cylindrical shell.

$$\overline{V} = \dot{u}\bar{i} + \dot{v}\bar{j} + \dot{w}\bar{k} + (\Omega\bar{i} \times w\bar{k}) + (\Omega\bar{i} \times v\bar{j}) \tag{3}$$

where $\dot{u}$, $\dot{v}$ and $\dot{w}$ are the components of the velocity in the x, θ, and z directions, respectively. The kinetic energy of the rotating shell is given by

$$T = \frac{1}{2}\rho h \int_0^L \int_0^{2\pi} \overline{V} \cdot \overline{V} R \, \mathrm{d}\theta \, \mathrm{d}x \tag{4}$$

Substituting Eq. (3) into Eq. (4), the kinetic energy expression of the shell can be expanded in the form

$$T = \frac{1}{2}\rho h \int_0^L \int_0^{2\pi} [\dot{u}^2 + \dot{v}^2 + \dot{w}^2 + 2\Omega(v\dot{w} - w\dot{v}) + \Omega^2(v^2 + w^2)] R \, \mathrm{d}\theta \, \mathrm{d}x \tag{5}$$

The initial hoop tension due to the centrifugal force is defined as

$$N_\theta = \rho h \Omega^2 R^2 \tag{6}$$

The strain energy of the shell due to hoop tension is

$$U_\mathrm{h} = \frac{1}{2} \int_0^L \int_0^{2\pi} N_\theta \left\{ \left(\frac{1}{R} \frac{\partial u}{\partial \theta} \right)^2 + \left[\frac{1}{R} \left(\frac{\partial v}{\partial \theta} + w \right) \right]^2 + \left[\frac{1}{R} \left(\frac{\partial w}{\partial \theta} - v \right) \right]^2 \right\} R \, \mathrm{d}\theta \, \mathrm{d}x \tag{7}$$

The strain energy of the shell is expressed as

$$U_\varepsilon = \frac{1}{2} \int_0^L \int_0^{2\pi} \varepsilon^\mathrm{T} [S] \varepsilon R \, \mathrm{d}\theta \, \mathrm{d}x \tag{8}$$

where $[S]$ is the stiffness matrix and the strain vector ε can be written as

$$\varepsilon^{\mathrm{T}} = \{ e_1 \quad e_2 \quad \gamma \quad \kappa_1 \quad \kappa_2 \quad 2\tau \} \tag{9}$$

where the middle surface strains, e_1, e_2 and γ, and the middle surface curvatures κ_1, κ_2 and τ, are defined according to Love's first approximation as follows

$$\begin{aligned} &e_1 = \frac{\partial u}{\partial x}, \qquad e_2 = \frac{1}{R}\left(\frac{\partial v}{\partial \theta} + w\right), \qquad \gamma = \frac{\partial v}{\partial x} + \frac{1}{R}\frac{\partial u}{\partial \theta}, \qquad \kappa_1 = -\frac{\partial^2 w}{\partial^2 x}, \\ &\kappa_2 = -\frac{1}{R^2}\left(\frac{\partial^2 w}{\partial^2 \theta} - \frac{\partial v}{\partial \theta}\right), \qquad \tau = -\frac{1}{R}\left(\frac{\partial^2 w}{\partial x \partial \theta} - \frac{\partial v}{\partial x}\right) \end{aligned} \tag{10}$$

and the stiffness matrix $[S]$ for a cross-ply laminated shell is given by

$$[S] = \begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0 \\ A_{12} & A_{22} & 0 & B_{12} & B_{22} & 0 \\ 0 & 0 & A_{66} & 0 & 0 & B_{66} \\ B_{11} & B_{12} & 0 & D_{11} & D_{12} & 0 \\ B_{12} & B_{22} & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & B_{66} & 0 & 0 & D_{66} \end{bmatrix} \tag{11}$$

where the extensional (A_{ij}), coupling (B_{ij}) and bending (D_{ij}) stiffnesses are defined as

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} Q_{ij}(1, z, z^2)\,\mathrm{d}z \tag{12}$$

For a shell composed of different layers of orthotropic material, the stiffnesses can be written as

$$A_{ij} = \sum_{k=1}^{N_1} \overline{Q}_{ij}^k (h_k - h_{k+1}), \qquad B_{ij} = \frac{1}{2}\sum_{k=1}^{N_1} \overline{Q}_{ij}^k (h_k^2 - h_{k+1}^2), \qquad D_{ij} = \frac{1}{3}\sum_{k=1}^{N_1} \overline{Q}_{ij}^k (h_k^3 - h_{k+1}^3) \tag{13}$$

where h_k and h_{k+1} denote the distances from the shell reference surface to the outer and inner surfaces of the kth layer as shown in Fig. 2. N_1 is the number of layers in the laminated shell. $\overline{Q}_{ij}^k$ is the transformed reduced stiffness matrix for the kth layer defined as

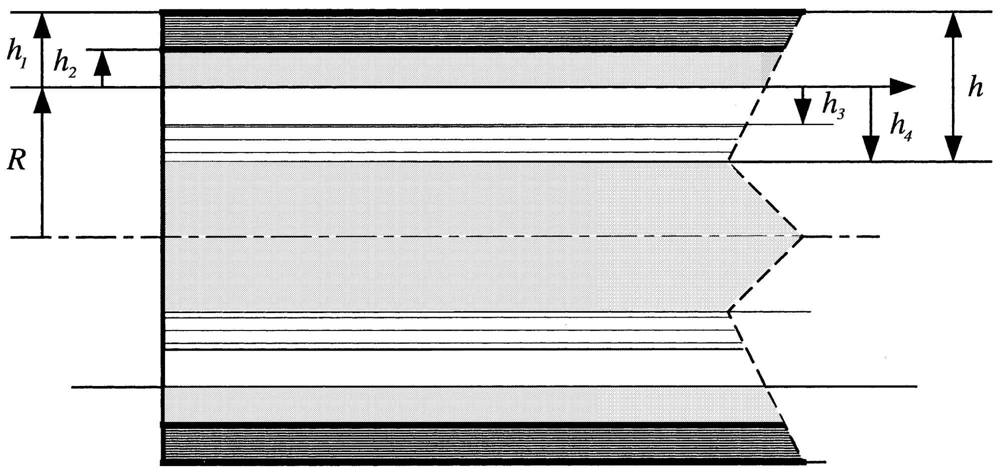

Fig. 2. Cross-sectional view of the laminated cylindrical shell.

$$[\overline{Q}] = [T]^{-1}[Q][T]^{-\mathrm{T}} \tag{14}$$

where $[T]$ is the transformation matrix for the principal material coordinates and the shell coordinates system and is defined as

$$[T] = \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & 2\cos\alpha\sin\alpha \\ \sin^2\alpha & \cos^2\alpha & -2\cos\alpha\sin\alpha \\ -\cos\alpha\sin\alpha & \cos\alpha\sin\alpha & \cos^2\alpha-\sin^2\alpha \end{bmatrix} \tag{15}$$

where α is the orientation of the fibers and $[Q]$ is the reduced stiffness matrix defined as

$$[Q] = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \tag{16}$$

and the material constants in the reduced stiffness matrix $[Q]$ are given as

$$Q_{11} = \frac{E_{11}}{1-v_{12}v_{21}}, \qquad Q_{12} = \frac{v_{12}E_{22}}{1-v_{12}v_{21}}, \qquad Q_{22} = \frac{E_{22}}{1-v_{12}v_{21}}, \qquad Q_{66} = G_{12} \tag{17}$$

where E_{11} and E_{22} are the elastic moduli, G_{12} the shear modulus, and v_{12} and v_{21} the Poisson's ratios. If the shell is assumed to be simply supported, the displacement components can be approximated as one-term expressions

$$u = A\cos\frac{m\pi x}{L}\cos(n\theta+\omega t), \qquad v = B\sin\frac{m\pi x}{L}\sin(n\theta+\omega t), \qquad w = C\sin\frac{m\pi x}{L}\cos(n\theta+\omega t) \tag{18}$$

where m represents the number of axial half waves, n the number of circumferential waves and ω the natural frequency of the rotating shell.

2.1. Stiffeners as discrete elements—single-term variational approximation

For the stiffeners, the displacements in the x, θ and z directions are defined as

$$u_{\mathrm{s}} = u - z\frac{\partial w}{\partial x}, \qquad v_{\mathrm{s}} = v - \frac{z}{R}\frac{\partial w}{\partial \theta}, \qquad w_{\mathrm{s}} = w \tag{19}$$

The strain of the stringers in the axial direction is described as

$$\varepsilon_{\mathrm{ss}} = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} \tag{20}$$

and strain of the rings in the circumferential direction is given by

$$\varepsilon_{\mathrm{rs}} = \frac{1}{R}\left(\frac{\partial v}{\partial \theta} - \frac{z}{R}\frac{\partial^2 w}{\partial \theta^2} + w\right) \tag{21}$$

The strain energy of the stringers can be expressed as

$$U_{\mathrm{s}} = \sum_{k=1}^{N_{\mathrm{s}}}\frac{1}{2}E_{\mathrm{s}k}\int_0^l\int_{A_{\mathrm{s}k}}\varepsilon_{\mathrm{ss}}^2\,\mathrm{d}A_{\mathrm{s}k}\,\mathrm{d}x + \sum_{k=1}^{N_{\mathrm{s}k}}\frac{1}{2}G_{\mathrm{s}k}J_{\mathrm{s}k}\int_0^l\frac{1}{R^2}\left(\frac{\partial^2 w}{\partial\theta\,\partial x}\right)^2\mathrm{d}x \tag{22}$$

where N_{s} is the number of the stringers, $E_{\mathrm{s}k}$, $A_{\mathrm{s}k}$ and $G_{\mathrm{s}k}J_{\mathrm{s}k}$ are the elastic modulus, cross-sectional area and torsional stiffness of the kth stringer, respectively.

The strain energy of the rings can be expressed as

$$U_{\mathrm{r}} = \sum_{k=1}^{N_{\mathrm{r}}} \frac{1}{2} E_{\mathrm{r}k} \int_0^l \int_{A_{\mathrm{r}k}} \varepsilon_{\mathrm{rs}}^2 \,\mathrm{d}A_{\mathrm{r}k}\,\mathrm{d}x + \sum_{k=1}^{N_{\mathrm{s}k}} \frac{1}{2} G_{\mathrm{r}k} J_{\mathrm{r}k} \int_0^{2\pi} \frac{1}{R^2} \left(\frac{\partial^2 w}{\partial\theta\,\partial x} \right)^2 R\,\mathrm{d}x \tag{23}$$

where N_{r} is the number of the rings, $E_{\mathrm{r}k}$, $A_{\mathrm{r}k}$ and $G_{\mathrm{r}k}J_{\mathrm{r}k}$ are the elastic modulus, cross-sectional area and torsional stiffness of kth ring, respectively. The kinetic energy of the stringers is given by

$$T_{\mathrm{s}} = \frac{1}{2}\rho \sum_{k=1}^{N_{\mathrm{s}}} \int_0^L \int_{A_{\mathrm{s}k}} [\dot{u}_{\mathrm{s}}^2 + \dot{v}_{\mathrm{s}}^2 + \dot{w}_{\mathrm{s}}^2 + 2\Omega(v_{\mathrm{s}}\dot{w}_{\mathrm{s}} - w_{\mathrm{s}}\dot{v}_{\mathrm{s}}) + \Omega^2(v_{\mathrm{s}}^2 + w_{\mathrm{s}}^2)]\,\mathrm{d}A_{\mathrm{s}k}\,\mathrm{d}x \tag{24}$$

and the kinetic energy of the rings is described as

$$T_{\mathrm{r}} = \frac{1}{2}\rho \sum_{k=1}^{N_{\mathrm{r}}} \int_0^{2\pi} \int_{A_{\mathrm{r}k}} [\dot{u}_{\mathrm{s}}^2 + \dot{v}_{\mathrm{s}}^2 + \dot{w}_{\mathrm{s}}^2 + 2\Omega(v_{\mathrm{s}}\dot{w}_{\mathrm{s}} - w_{\mathrm{s}}\dot{v}_{\mathrm{s}}) + \Omega^2(v_{\mathrm{s}}^2 + w_{\mathrm{s}}^2)]R\,\mathrm{d}A_{\mathrm{r}k}\,\mathrm{d}\theta \tag{25}$$

The strain energy of the rings due to hoop tension is taken to be

$$U_{\mathrm{rh}} = \frac{1}{2} N_\theta \sum_{k=1}^{N_{\mathrm{r}}} \int_0^{2\pi} \int_{A_{\mathrm{r}k}} \left\{ \left(\frac{1}{R}\frac{\partial u_{\mathrm{s}}}{\partial\theta} \right)^2 + \left[\frac{1}{R}\left(\frac{\partial v_{\mathrm{s}}}{\partial\theta} + w_{\mathrm{s}} \right) \right]^2 + \left[\frac{1}{R}\left(\frac{\partial w_{\mathrm{s}}}{\partial\theta} - v_{\mathrm{s}} \right) \right]^2 \right\} R\,\mathrm{d}A_{\mathrm{r}k}\,\mathrm{d}\theta \tag{26}$$

The energy functional can thus be written as

$$\Pi = T + T_{\mathrm{r}} + T_{\mathrm{s}} - U_{\mathrm{a}} - U_{\mathrm{h}} - U_{\varepsilon} - U_{\mathrm{s}} - U_{\mathrm{r}} - U_{\mathrm{rh}} \tag{27}$$

Substituting Eq. (18) into Eq. (27) and applying variational principles

$$\delta \int_t^{t+2\pi/\omega} \Pi\,\mathrm{d}t = 0 \tag{28}$$

the following matrix relationship can be established

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = 0 \tag{29}$$

where the coefficients α_{ij} $(i, j = 1, 2, 3)$ are given in detail in Appendix A. For non-trivial solutions of Eq. (29)

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix} = 0 \tag{30}$$

Expanding Eq. (30), the characteristic frequency equation can be obtained as

$$\beta_1\omega_{mn}^6 + \beta_2\omega_{mn}^5 + \beta_3\omega_{mn}^4 + \beta_4\omega_{mn}^3 + \beta_5\omega_{mn}^2 + \beta_6\omega + \beta_7 = 0 \tag{31}$$

It is should be noted the coefficient for the term ω^5 will be zero for unstiffened rotating shells.

2.2. Averaging method—stiffener properties averaged over shell surface

If the stiffener spacing is sufficiently small, the effects of the stiffeners can be averaged over the domain of the shell. The strain energy of the stringers in this case can thus be expressed as

$$U'_{\mathrm{s}} = \frac{1}{d}\int_0^{2\pi}\int_0^L \left[\frac{E_{\mathrm{s}}}{2}\int_{A_{\mathrm{s}}} \varepsilon_{\mathrm{ss}}^2 \,\mathrm{d}A_{\mathrm{s}} + \frac{G_{\mathrm{s}}J_{\mathrm{s}}}{2R^2} w_{,x\theta}^2\right] R\,\mathrm{d}x\,\mathrm{d}\theta \tag{32}$$

where d is the stringer spacing, E_{s}, A_{s} and $G_{\mathrm{s}}J_{\mathrm{s}}$ are the elastic modulus, cross-sectional area and torsional stiffness of the stringers, respectively. The strain energy of the rings can be expressed as

$$U'_{\mathrm{r}} = \frac{1}{l}\int_0^{2\pi}\int_0^L \left[\frac{E_{\mathrm{r}}}{2}\int_{A_{\mathrm{r}}} \varepsilon_{\mathrm{rs}}^2 \,\mathrm{d}A_{\mathrm{r}} + \frac{G_{\mathrm{r}}J_{\mathrm{r}}}{2R^2} w_{,x\theta}^2\right] R\,\mathrm{d}x\,\mathrm{d}\theta \tag{33}$$

where l is the ring spacing, E_{r}, A_{r} and $G_{\mathrm{r}}J_{\mathrm{r}}$ are the elastic modulus, cross-sectional area and torsional stiffness of the rings, respectively. The kinetic energy of the stringers is thus

$$T'_{\mathrm{s}} = \frac{1}{2d}\rho\int_0^{2\pi}\int_0^L\int_{A_{\mathrm{s}}} [\dot{u}_{\mathrm{s}}^2 + \dot{v}_{\mathrm{s}}^2 + \dot{w}_{\mathrm{s}}^2 + 2\Omega(v_{\mathrm{s}}\dot{w}_{\mathrm{s}} - w_{\mathrm{s}}\dot{v}_{\mathrm{s}}) + \Omega^2(v_{\mathrm{s}}^2 + w_{\mathrm{s}}^2)]R\,\mathrm{d}A_{\mathrm{s}}\,\mathrm{d}x\,\mathrm{d}\theta \tag{34}$$

The kinetic energy of the rings is expressed as

$$T'_{\mathrm{r}} = \frac{1}{2l}\rho\int_0^{2\pi}\int_0^L\int_{A_{\mathrm{r}}} [\dot{u}_{\mathrm{s}}^2 + \dot{v}_{\mathrm{s}}^2 + \dot{w}_{\mathrm{s}}^2 + 2\Omega(v_{\mathrm{s}}\dot{w}_{\mathrm{s}} - w_{\mathrm{s}}\dot{v}_{\mathrm{s}}) + \Omega^2(v_{\mathrm{s}}^2 + w_{\mathrm{s}}^2)]R\,\mathrm{d}A_{\mathrm{r}}\,\mathrm{d}x\,\mathrm{d}\theta \tag{35}$$

The strain energy of the stringers due to the hoop tension is

$$U'_{\mathrm{rh}} = \frac{1}{2}N_\theta\int_0^{2\pi}\int_0^L\int_{A_{\mathrm{r}}} \left\{\left(\frac{1}{R}\frac{\partial u_{\mathrm{s}}}{\partial\theta}\right)^2 + \left[\frac{1}{R}\left(\frac{\partial v_{\mathrm{s}}}{\partial\theta} + w_{\mathrm{s}}\right)\right]^2 + \left[\frac{1}{R}\left(\frac{\partial w_{\mathrm{s}}}{\partial\theta} + v_{\mathrm{s}}\right)\right]^2\right\} R\,\mathrm{d}A_{\mathrm{r}}\,\mathrm{d}\theta\,\mathrm{d}x \tag{36}$$

Thus the energy functional can be written as

$$\Pi' = T + T'_{\mathrm{s}} + T'_{\mathrm{r}} - U_{\mathrm{a}} - U_{\mathrm{h}} - U_{\varepsilon} - U'_{\mathrm{s}} - U'_{\mathrm{r}} - U'_{\mathrm{rh}} \tag{37}$$

and according to the Rayleigh–Ritz minimization procedure

$$\frac{\partial\Pi'}{\partial\Delta} = 0, \quad \Delta = A, B, C \tag{38}$$

The following relationship can be established in matrix form

$$\begin{bmatrix} \alpha'_{11} & \alpha'_{12} & \alpha'_{13} \\ \alpha'_{21} & \alpha'_{22} & \alpha'_{23} \\ \alpha'_{31} & \alpha'_{32} & \alpha'_{33} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{39}$$

where the coefficients α'_{ij} $(i, j = 1, 2, 3)$ are given in detail in Appendix A. For non-trivial solutions, the determinant of the characteristic matrix in Eq. (39) is equated to zero

$$\begin{vmatrix} \alpha'_{11} & \alpha'_{12} & \alpha'_{13} \\ \alpha'_{21} & \alpha'_{22} & \alpha'_{23} \\ \alpha'_{31} & \alpha'_{32} & \alpha'_{33} \end{vmatrix} = 0 \tag{40}$$

Expanding Eq. (40), the characteristic frequency equation in similar form as that of Eq. (31) can be obtained as

$$\beta'_1\omega_{mn}^6 + \beta'_2\omega_{mn}^5 + \beta'_3\omega_{mn}^4 + \beta'_4\omega_{mn}^3 + \beta'_5\omega_{mn}^2 + \beta'_6\omega + \beta'_7 = 0 \tag{41}$$

Eqs. (31) and (41) can be solved by the Newton–Raphson's method.

3. Results and discussions

The material properties of the stiffeners used in the present study are listed in Table 1. Table 2 presents the frequency parameters of non-rotating cross-ply laminated cylindrical shells with different combination of stringer/ring stiffeners via the two methods proposed, i.e., a variational formulation treating individual stiffeners as discrete elements, and an averaging method in which the properties of the stiffeners are averaged over the shell surface. The lamination scheme is [0°/90°/0°], the shell material property parameters used are $E_{11}/E_{22} = 2.5$ and $\nu_{12} = 0.26$, and the shell geometry parameters are $L/R = 4$ and $h/R = 0.005$. Numerical results are generated and compared to examine the accuracy of these two approaches. Table 2 shows that there exist significant frequency discrepancies between these two methods when the combination of stringer/ring stiffeners is 2/2. This discrepancy becomes significantly smaller when the stringer/ring combination is increased to 4/4. The results from these two methods of analysis converge when the stringer/ring combination is increased to 10/5. This comparison study clearly shows that the averaging method has inherent errors when the number of the stiffeners is not sufficient. All subsequent current results presented utilize the variational formulation where the contribution of each stiffener is treated individually in deriving the global energy functional.

For unstiffened rotating cross-ply laminated cylindrical shells with lamination scheme of [0°/90°/0°] and length ratios $L/R = 1$, results are compared with those of Lam and Loy (1995) for the verification of the validity of the present formulation. The comparisons are presented in Table 3, in which ω^* is the non-dimensional frequency parameter, and ω^*_{f} and ω^*_{b} are the non-dimensional frequency parameters associated with the forward and backward waves respectively. At various speeds of revolution and circumferential

Table 1
The geometrical parameters and material properties of the stiffeners used in the present study

Stiffener type	Stringer (external)	Ring (external)
Depth (m)	0.008	0.008
Width (m)	0.002	0.002
E (N/m^2)	3.0E_{11}	3.0E_{11}
ν	0.3	0.3
ρ (kg/m^3)	1643	1643

Table 2
Natural frequencies of a [0°/90°/0°] cross-ply laminated cylindrical shell, axial wave number $m = 1$, $L/R = 4$, $h/R = 0.005$, $E_{11}/E_{22} = 2.5$ and $\nu_{12} = 0.26$

Circumferential wave number (n)	Stringers/rings configuration					
	2/2		4/4		10/5	
	Variational	Average	Variational	Average	Variational	Average
1	568.0	561.2	561.4	553.7	549.8	549.7
2	278.1	280.9	286.4	288.1	299.9	299.8
3	216.2	255.8	261.4	291.6	305.1	305.2
4	306.7	398.4	404.5	470.6	486.6	486.7
5	469.4	619.3	627.5	734.6	756.8	756.9
6	676.9	895.6	906.9	1062.8	1093.7	1093.8
7	924.1	1223.3	1238.4	1451.5	1493.0	1493.0
8	1209.7	1601.5	1620.9	1899.7	1953.3	1953.3
9	1533.3	2029.6	2053.9	2406.9	2474.1	2474.1
10	1894.7	2507.7	2537.4	2972.8	3054.9	3054.9

Table 3
Non-dimensional frequency parameter $\omega^* = \omega(\rho R^2/E_{22})^{1/2}$ for a [0°/90°/0°] simply supported rotating laminated cylindrical shell ($h/R = 0.002$, $L/R = 1$)

Ω (rev/s)	n	Lam and Loy (1995)		Present (variational)	
		ω_b^*	ω_f^*	ω_b^*	ω_f^*
0.1	1	1.061429	1.061140	1.061428	1.061139
	2	0.804214	0.803894	0.804212	0.803892
	3	0.598476	0.598157	0.598472	0.598183
	4	0.450270	0.450021	0.450263	0.450015
	5	0.345363	0.345149	0.345355	0.345140
	6	0.270852	0.270667	0.270840	0.270654
	7	0.217651	0.217489	0.217635	0.217473
0.4	1	1.061862	1.060706	1.061862	1.060705
	2	0.804696	0.803415	0.804694	0.803413
	3	0.598915	0.597762	0.598911	0.597758
	4	0.450662	0.449667	0.450654	0.449660
	5	0.345724	0.344870	0.345714	0.344860
	6	0.271207	0.270468	0.271193	0.270454
	7	0.218029	0.217382	0.218011	0.217364
1.0	1	1.062728	1.059836	1.062728	1.059837
	2	0.805667	0.802464	0.805664	0.802461
	3	0.599820	0.596937	0.599813	0.596930
	4	0.451513	0.449027	0.451502	0.449015
	5	0.346593	0.344459	0.346577	0.344442
	6	0.272197	0.270349	0.272174	0.270326
	7	0.219269	0.217651	0.219240	0.217621

wave numbers, very good agreement is observed. For stiffened non-rotating isotropic cylindrical shells, comparison of present results are made with those of Mustafa and Ali (1989), Egle and Sewall (1968), Rinehart and Wang (1972) and an ESDU (1982) report. The geometric and material properties used in each of these works for comparison are given in Table 4. The comparison with existing results are presented in Tables 5–7 and very good agreement is generally observed.

Figs. 3 and 4 respectively illustrate the effects of the number of stringers and rings on the frequencies of the rotating shells for mode $(1, 3)$. In this study, the stringers and rings are equally spaced. It is observed that the frequencies of the shells generally increase with the increase of the number of the stringers and rings

Table 4
Geometrical and material properties of stiffened shells used for comparison

	Egle and Sewall (1968)	Rinehart and Wang (1972)	ESDU (1982)
Number of stiffeners	60	4	13/20
Shell radius (m)	0.242	0.1945	0.203
Shell thickness (m)	0.65×10^{-3}	0.464×10^{-3}	2.04×10^{-3}
Shell length (m)	0.6096	0.9868	0.813
Stiffener depth (m)	0.00702	0.0101	0.006/0.006
Stiffener width (m)	0.002554	0.00104	0.004/0.008
E (N/m^2)	68.95×10^9	200×10^9	207×10^9
ν	0.3	0.3	0.3
ρ (kg/m^3)	2714	7770	7430
Stiffener type	Stringer (external)	Stringer (internal)	Ring/stringer (internal)

Table 5
Comparison of natural frequencies of a shell stiffened with 60 external stringers, axial wave number $m = 1$

Circumferential wave number (n)	Natural frequencies (Hz)		
	Present (variational)	Egle and Sewall (1968)	Mustafa and Ali (1989)
1	1147	–	1141
2	676	–	674
3	429	–	427
4	299	–	296
5	231	231	225
6	198	197	188
7	189	189	174
8	196	199	177
9	214	219	193
10	241	252	218

Table 6
Comparison of natural frequencies of a shell stiffened with four internal stringers, axial wave number $m = 1$

Circumferential wave number (n)	Natural frequencies (Hz)		
	Present (variational)	Rinehart and Wang (1972)	Mustafa and Ali (1989)
1	777.5	–	778
2	316.8	–	317
3	159	156	159
4	101	100	99.6
5	91	89	91.5
6	109	104	106
7	141	137	142
8	183	174	178
9	231	224	231
10	285	265	277

Table 7
Comparison of natural frequencies of a shell stiffened with 60 external stringers, axial wave number $m = 1$

Circumferential wave number (n)	Natural frequencies (Hz)		
	Present (variational)	ESDU (1982)	Mustafa and Ali (1989)
1	930	938	942
2	442	443	439
3	359	348	337
4	518	492	482
5	785	745	740

for both the forward and backward waves. However, it is also seen in Fig. 3 that the frequencies of the shell show very minimal increase when the stringer number is increased from 50 to 60. It is thus expected that the frequencies of the rotating shell will converge when the number of stringers is sufficiently high. Trends toward frequency convergence when the number of rings is sufficiently high are also observed from Fig. 4. Fig. 5 depicts the effects of variation of stringer/ring combination on the forward and backward travelling waves of mode $(1, 3)$ for the rotating stiffened cylindrical shells. In this case, the number of the stringers is 5 and the number of the rings is varied from 1 to 11. It is observed that the frequencies of the shell increase

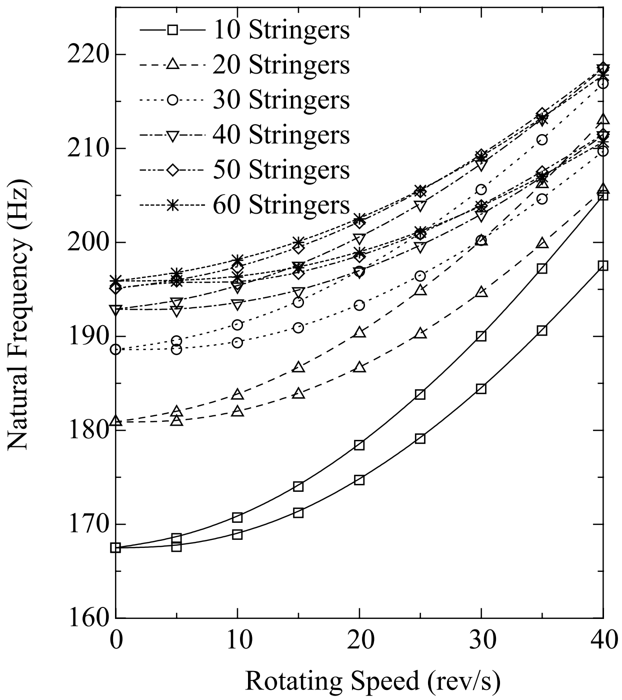

Fig. 3. Variation of the bifurcation of natural frequencies, mode $(m, n) = (1, 3)$, for a rotating composite laminated cylindrical shell with different number of stringers. Lamination scheme [0°/90°/0°], $L/R = 4$, $h/R = 0.005$, $E_{11}/E_{22} = 2.5$ and $v_{12} = 0.26$.

due to the increase of ring numbers. However, it is noted that the sensitivity of the frequencies of the shell to the number of rings decreases with the increment of the ring numbers.

Figs. 6–8 illustrate the effects of the depth to width ratio of the stiffeners on the frequencies for the axial, circumferential and transverse modes, respectively. The results are based on a rotating speed of 10 rev/s, stringer/ring combination of 10/5, and modes $(1, 2)$, $(1, 3)$ and $(1, 4)$ are presented. Fig. 6 shows that the frequencies of the axial mode increase with increases in the depth-to-width ratios of the stiffeners for all three modes considered. It is also observed that increases in the magnitudes of the frequencies with the depth-to-width ratios of the stiffeners are more or less similar for each of the three modes. Fig. 7 shows that the circumferential mode frequencies of the shell decrease with increase of the depth-to-width ratio for modes $(1, 3)$ and $(1, 4)$. For mode $(1, 2)$ however, the depth to width ratios of the stiffeners have almost no effects on the frequencies of the shell. Fig. 8 reveals that the depth-to-width ratio of the stiffeners significantly affect the transverse mode frequencies of the shell. It is observed that the frequencies significantly increase with the increase of the depth-to-width ratio for both the forward and backward waves. It is also observed that the effects of increase in the depth-to-width ratios of the stiffeners on the transverse mode frequencies of the shell are more pronounced for higher circumferential modes.

4. Conclusions

The free vibration analysis of the stiffened simply supported rotating cross-ply laminated cylindrical shells has been investigated via an energy approach. The accuracy of two methods in treating the stringer/ring stiffeners is examined. It was found that the averaging method, where the properties of the stiffeners are

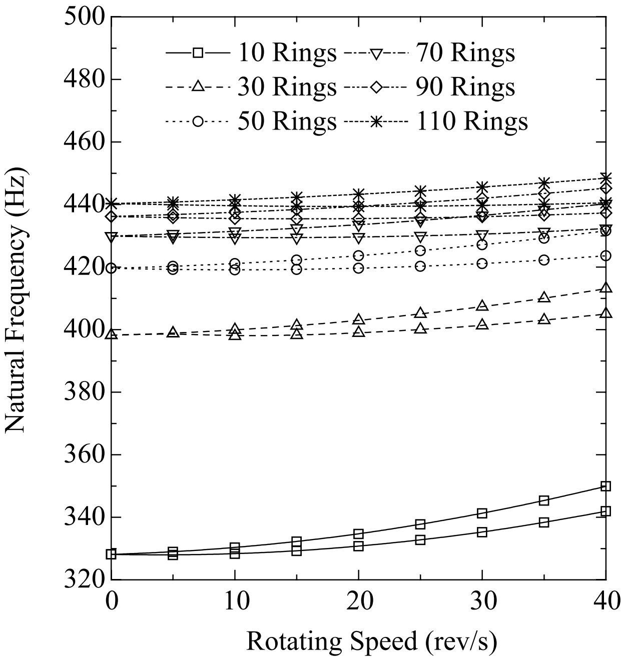

Fig. 4. Variation of the bifurcation of natural frequencies, mode $(m, n) = (1, 3)$, for a rotating composite laminated cylindrical shell with different number of rings. Lamination scheme $[0°/90°/0°]$, $L/R = 4$, $h/R = 0.005$, $E_{11}/E_{22} = 2.5$ and $v_{12} = 0.26$.

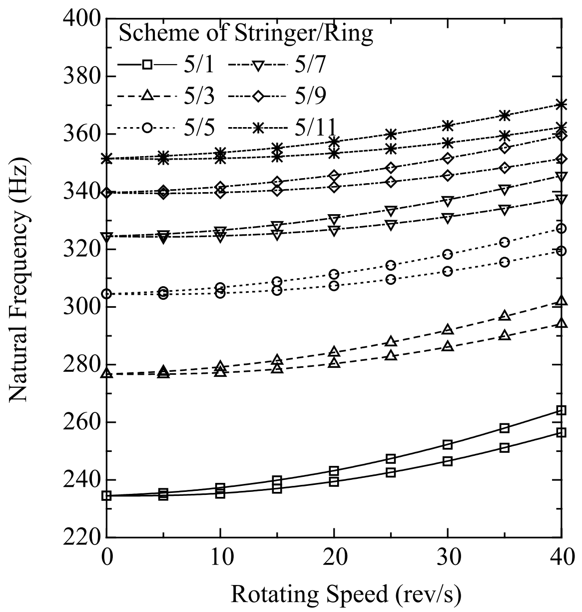

Fig. 5. Variation of the bifurcation of natural frequencies, mode $(m, n) = (1, 3)$, for a rotating composite laminated cylindrical shell with different stringer/ring schemes. Lamination scheme $[0°/90°/0°]$, $L/R = 4$, $h/R = 0.005$, $E_{11}/E_{22} = 2.5$ and $v_{12} = 0.26$.

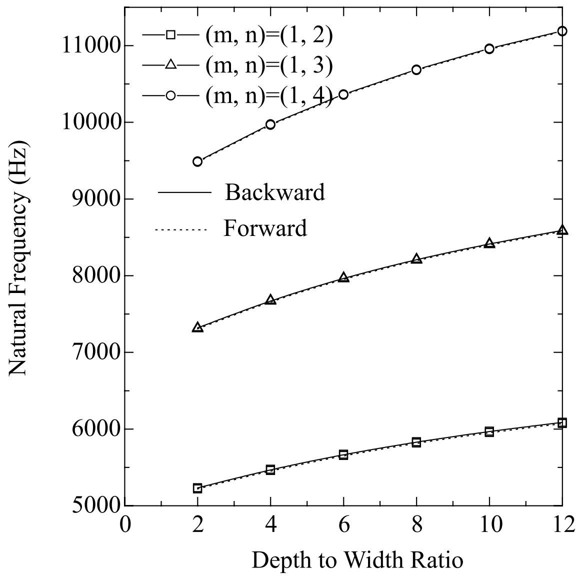

Fig. 6. Variation of the axial mode frequencies for a rotating composite laminated cylindrical shell with different stiffener depth to width ratios. Lamination scheme [0°/90°/0°], stringer/ring ~ 10/5, $L/R = 4$, $h/R = 0.005$, $E_{11}/E_{22} = 2.5$ and $\nu_{12} = 0.26$, $\Omega = 10$ rev/s.

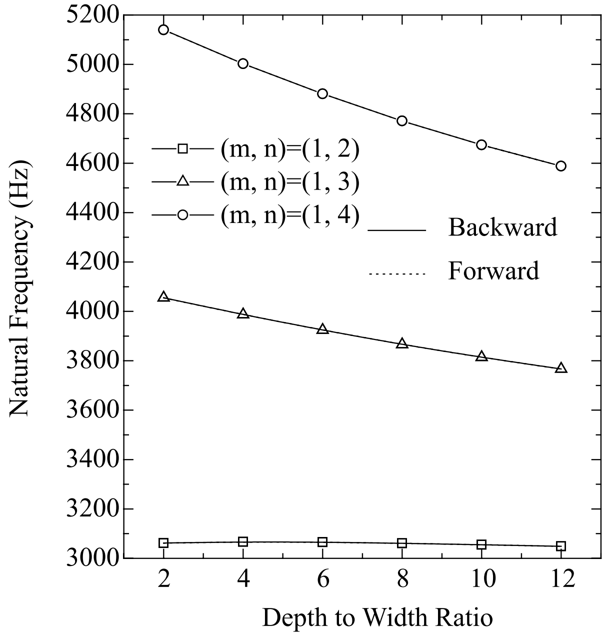

Fig. 7. Variation of the circumferential mode frequencies for a rotating composite laminated cylindrical shell with different stiffener depth to width ratios. Lamination scheme [0°/90°/0°], stringer/ring ~ 10/5, $L/R = 4$, $h/R = 0.005$, $E_{11}/E_{22} = 2.5$ and $\nu_{12} = 0.26$, $\Omega = 10$ rev/s.

averaged over the shell surface, is especially sensitive to the overall number of stiffeners and produces inaccurate results when the number of stiffeners is small. The variational formulation, in which the

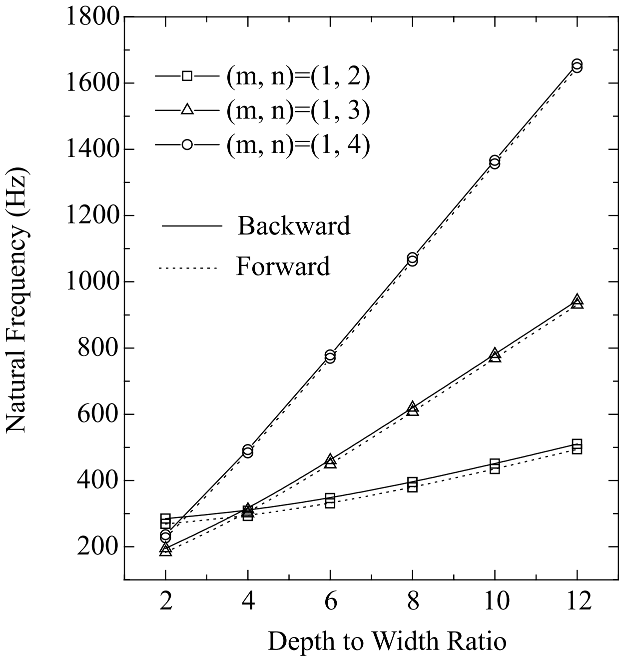

Fig. 8. Variation of the transverse mode frequencies for a rotating composite laminated cylindrical shell with different stiffener depth to width ratios. Lamination scheme [0°/90°/0°], stringer/ring ∼ 10/5, $L/R = 4$, $h/R = 0.005$, $E_{11}/E_{22} = 2.5$ and $v_{12} = 0.26$, $\Omega = 10$ rev/s.

contribution of each stiffener is treated individually, is much more robust and elegant, at the same time not requiring that much more computational effort. The effects of the number of orthogonal stringers and rings as well as the depth-to-width ratios of the stiffeners, on the frequencies of the shells, have also been examined. It is noted that, in general, the frequencies of the shell increase with increased number of stiffeners. It is also found that the depth-to-width ratios of stiffeners have significantly higher effects on the transverse mode frequencies compared with the axial and circumferential modes.

Appendix A

$$\alpha_{11} = L\pi^2\left(\rho hR\omega^2 - N_a R\lambda^2 - \rho hR\Omega^2 n^2 - \frac{A_{66}n^2}{R} - A_{11}R\lambda^2\right) - \frac{1}{2}N_s L\pi\lambda^2 A_s(E_s + N_a)$$
$$- \frac{2.0n^2\pi^2 A_r N_\theta}{R}\sum_{k=1}^{N_r}\cos^2\lambda x_k + \left(2.0\pi^2\rho R A_r\sum_{k=1}^{N_r}\cos^2\lambda x_k + \frac{1}{2}N_s\rho A_s L\pi\right)\omega^2$$

where $\lambda = m\pi/L$.

$$\alpha_{12} = L\pi^2 n\lambda\left(A_{12} + A_{66} + \frac{B_{12}}{R} + \frac{2B_{66}}{R}\right)$$

$$\alpha_{13} = L\pi^2\lambda\left[A_{12} + B_{11}R\lambda^2 + \frac{B_{12}n^2}{R} + \frac{2B_{66}n^2}{R}\right] + \frac{2.0n^2\pi^2\lambda A_r N_\theta z_r}{R}\sum_{k=1}^{N_r}\cos^2\lambda x_k$$
$$+ \frac{1}{2}LN_s\pi\lambda^3 A_s z_s(E_s + N_a) - \left(2.0\pi^2 R\lambda\rho A_r z_r\sum_{k=1}^{N_r}\cos^2\lambda x_k + \frac{1}{2}LN_s\pi\lambda\rho A_s z_s\right)\omega^2$$

$$\alpha_{21} = \alpha_{12}$$

$$\begin{aligned}\alpha_{22} = {} & L\pi^2\left(\rho hR\omega^2 - N_\mathrm{a}R\lambda^2 - \rho hn^2\Omega^2R - \frac{D_{22}n^2}{R^3} - \frac{2B_{22}n^2}{R^2} - \frac{A_{22}n^2}{R} - \frac{4D_{66}\lambda^2}{R} - 4B_{66}\lambda^2 - A_{66}\lambda^2R\right) \\ & + 2\pi^2R\rho A_\mathrm{r}\Omega^2\sum_{k=1}^{N_\mathrm{r}}\sin^2\lambda x_k + \frac{1}{2}LN_\mathrm{s}\pi\rho A_\mathrm{s}\Omega^2 - \frac{2n^2\pi^2A_\mathrm{r}E_\mathrm{r}}{R}\sum_{k=1}^{N_\mathrm{r}}\sin^2\lambda x_k - \frac{1}{2}LN_\mathrm{s}\pi\lambda_\mathrm{s}^2A_\mathrm{s}N_\mathrm{a} \\ & - \frac{2\pi^2A_\mathrm{r}N_\theta(n^2+1)}{R}\sum_{k=1}^{N_\mathrm{r}}\sin^2\lambda x_k + \left(2\pi^2R\rho A_\mathrm{r}\sum_{k=1}^{N_\mathrm{r}}\sin^2\lambda x_k + \frac{1}{2}LN_\mathrm{s}\rho\pi A_\mathrm{s}\right)\omega^2\end{aligned}$$

$$\begin{aligned}\alpha_{23} = {} & L\pi^2n\left(-\frac{A_{22}}{R} - B_{12}\lambda^2 - \frac{B_{22}n^2}{R^2} - 2B_{66}\lambda^2 - \frac{B_{22}}{R^3} - \frac{D_{22}n^2}{R^3} - \frac{D_{12}\lambda^2}{R} - \frac{4D_{66}\lambda^2}{R} - 2\rho hR\Omega^2\right) \\ & - \frac{2n\pi^2A_\mathrm{r}(E_\mathrm{r}+2N_\theta)}{R}\sum_{k=1}^{N_\mathrm{r}}\sin^2\lambda x_k + 2n^2\pi\rho\Omega^2A_\mathrm{r}z_\mathrm{r}\sum_{k=1}^{N_\mathrm{r}}\sin^2\lambda x_k + \frac{LN_\mathrm{s}n\pi A_\mathrm{s}z_\mathrm{s}(\rho\Omega^2-\lambda^2N_\mathrm{a})}{2R} \\ & - \frac{2n\pi^2A_\mathrm{r}z_\mathrm{r}(n^2E_\mathrm{r}+N_\theta+n^2N_\theta)}{R^2}\sum_{k=1}^{N_\mathrm{r}}\sin^2\lambda x_k + \left(2n\pi^2\rho A_\mathrm{r}z_\mathrm{r}\sum_{k=1}^{N_\mathrm{r}}\sin^2\lambda x_k + \frac{LnN_\mathrm{s}\pi\rho A_\mathrm{s}z_\mathrm{s}}{2R}\right)\omega^2 \\ & - (4\pi^2\rho\Omega RA_\mathrm{r} + LN_\mathrm{s}\pi\rho\Omega A_\mathrm{s} + 2L\pi^2\rho hR\Omega)\omega\end{aligned}$$

$$\alpha_{31} = \alpha_{13}, \qquad \alpha_{32} = \alpha_{23}$$

$$\begin{aligned}\alpha_{33} = {} & L\pi^2\left[\rho hR(\omega^2+\Omega^2) - N_\mathrm{a}\lambda^2R - \rho hn^2\Omega^2R - 2B_{12}\lambda^2 - \frac{D_{22}}{R^3} - \frac{2D_{22}n^2}{R^3} - \frac{2B_{22}}{R^2} - \frac{2B_{22}}{R^2} - \frac{D_{22}n^4}{R^3} - \frac{A_{22}}{R}\right. \\ & \left. - \frac{2D_{12}\lambda^2}{R} - \frac{2D_{12}n^2\lambda^2}{R} - \frac{4D_{66}n^2\lambda^2}{R} - D_{11}\lambda^4R\right] + \left(\frac{2n^2\pi^2\rho\Omega^2I_\mathrm{or}}{2R^2} + 2\pi^2R\rho\Omega^2A_\mathrm{r}\right)\sum_{k=1}^{N_\mathrm{r}}\sin^2\lambda x_k \\ & + \frac{1}{2}LN_\mathrm{s}\pi\rho\Omega^2A_\mathrm{s} - \left(\frac{2n^4\pi^2I_\mathrm{or}E_\mathrm{r}}{R^3} + \frac{2\pi^2A_\mathrm{r}E_\mathrm{r}}{R}\right)\sum_{k=1}^{N_\mathrm{r}}\sin^2\lambda x_k - \frac{2n^2\pi^2\lambda^2(G_\mathrm{r}J_\mathrm{r}+I_\mathrm{or}N_\theta)}{R}\sum_{k=1}^{N_\mathrm{r}}\cos^2\lambda x_k \\ & - \frac{1}{2}LN_\mathrm{s}\pi\lambda^4I_\mathrm{os}(E_\mathrm{s}+N_\mathrm{a}) - \frac{Ln^2N_\mathrm{s}\pi\lambda^2}{2R^2}(G_\mathrm{s}J_\mathrm{s}+I_\mathrm{os}N_\mathrm{a}) - \frac{2n^2\pi^2I_\mathrm{or}N_\theta(1+n^2)}{R^3}\sum_{k=1}^{N_\mathrm{r}}\sin^2\lambda x_k \\ & - \frac{1}{2}LN_\mathrm{s}\pi\lambda^2A_\mathrm{s}N_\mathrm{a} - \frac{2\pi^2A_\mathrm{r}N_\theta(n^2+1)}{R}\sum_{k=1}^{N_\mathrm{r}}\sin^2\lambda x_k - \frac{4n^2\pi^2A_\mathrm{r}z_\mathrm{r}(E_\mathrm{r}+2N_\theta)}{R^2}\sum_{k=1}^{N_\mathrm{r}}\sin^2\lambda x_k \\ & + \left\{LN_\mathrm{s}\pi\rho I_\mathrm{os}\left(\frac{n^2}{2R^2}+\frac{1}{2}\lambda^2\right) + 2\pi^2\rho\left[\left(\frac{n^2I_\mathrm{or}}{R}+RA_\mathrm{r}\right)\sum_{k=1}^{N_\mathrm{r}}\sin^2\lambda x_k + R\lambda^2I_\mathrm{or}\sum_{k=1}^{N_\mathrm{r}}\cos^2\lambda x_k\right]\right. \\ & \left. + \frac{1}{2}LN_\mathrm{s}\pi\rho A_\mathrm{s}\right\}\omega^2 - \left(8n\pi^2\rho\Omega A_\mathrm{r}z_\mathrm{r}\sum_{k=1}^{N_\mathrm{r}}\sin^2\lambda x_k + \frac{2LnN_\mathrm{s}\pi\rho\Omega A_\mathrm{s}z_\mathrm{s}}{R}\right)\omega\end{aligned}$$

$$\begin{aligned}\alpha'_{11} = {} & \frac{1}{2}L\pi\left(\rho hR\omega^2 - N_\mathrm{a}R\lambda^2 - \rho hR\Omega^2n^2 - \frac{A_{66}n^2}{R} - A_{11}R\lambda^2\right) - \frac{1}{2d}L\pi R\lambda^2A_\mathrm{s}(E_\mathrm{s}+N_\mathrm{a}) - \frac{Ln^2\pi A_\mathrm{r}N_\theta}{2Rl} \\ & + \frac{1}{2}L\pi R\rho\left(\frac{A_\mathrm{r}}{l}+\frac{A_\mathrm{s}}{d}\right)\omega^2\end{aligned}$$

$$\alpha'_{12} = \frac{1}{2}L\pi n\lambda\left(A_{12} + A_{66} + \frac{B_{12}}{R} + \frac{2B_{66}}{R}\right)$$

$$\alpha'_{13} = \frac{1}{2}L\pi\lambda\left[A_{12} + B_{11}R\lambda^2 + \frac{B_{12}n^2}{R} + \frac{2B_{66}n^2}{R}\right] + \frac{Ln^2\pi\lambda A_r N_\theta z_r}{2Rl} + \frac{1}{2d}LR\pi\lambda^3 A_s z_s(E_s + N_a)$$
$$- L\pi R\lambda\rho\left(\frac{1}{2d}A_r z_r + \frac{1}{2l}A_s z_s\right)\omega^2$$

$$\alpha'_{21} = \alpha'_{12}$$

$$\alpha'_{22} = \frac{1}{2}L\pi\left(\rho hR\omega^2 - N_a R\lambda^2 - \rho hn^2\Omega^2 R - \frac{D_{22}n^2}{R^3} - \frac{2B_{22}n^2}{R^2} - \frac{A_{22}n^2}{R} - \frac{4D_{66}\lambda^2}{R} - 4B_{66}\lambda^2 - A_{66}\lambda^2 R\right)$$
$$+ \frac{1}{2}L\pi R\rho\Omega^2\left(\frac{A_r}{l} + \frac{A_s}{d}\right) - \frac{Ln^2\pi A_r E_r}{2Rl} - \frac{1}{2d}LR\pi\lambda^2 A_s N_a - \frac{L\pi^2 A_r N_\theta(n^2+1)}{2Rl} + \frac{1}{2}LR\rho\pi(A_r + A_s)\omega^2$$

$$\alpha'_{23} = \frac{1}{2}L\pi n\left(-\frac{A_{22}}{R} - B_{12}\lambda^2 - \frac{B_{22}n^2}{R^2} - 2B_{66}\lambda^2 - \frac{B_{22}}{R^3} - \frac{D_{22}n^2}{R^3} - \frac{D_{12}\lambda^2}{R} - \frac{4D_{66}\lambda^2}{R} - 2\rho hR\Omega^2\right)$$
$$- \frac{Ln\pi A_r(E_r + 2N_\theta)}{2Rl} + \frac{Ln\pi\rho\Omega^2 A_r z_r}{2l} + \frac{Ln\pi A_s z_s(\rho\Omega^2 - \lambda^2 N_a)}{2d} - \frac{Ln\pi A_r z_r(n^2 E_r + N_\theta + n^2 N_\theta)}{2R^2 l}$$
$$+ \frac{1}{2}Ln\pi\rho\left(\frac{A_r z_r}{l} + \frac{A_s z_s}{d}\right)\omega^2 - L\pi\rho\Omega R\left(\frac{A_r}{l} + \frac{A_s}{d} + h\right)\omega$$

$$\alpha'_{31} = \alpha'_{13}, \qquad \alpha'_{32} = \alpha'_{23}$$

$$\alpha'_{33} = \frac{1}{2}L\pi\left[\rho hR(\omega^2 + \Omega^2) - N_0\lambda^2 R - \rho hn^2\Omega^2 R - 2B_{12}\lambda^2 - \frac{D_{22}}{R^3} - \frac{2D_{22}n^2}{R^3} - \frac{2B_{22}}{R^2} - \frac{2B_{22}}{R^2} - \frac{D_{22}n^4}{R^3} - \frac{A_{22}}{R}\right.$$
$$\left. - \frac{2D_{12}\lambda^2}{R} - \frac{2D_{12}n^2\lambda^2}{R} - \frac{4D_{66}n^2\lambda^2}{R} - D_{11}\lambda^4 R\right] + Ln^2\pi\rho\Omega^2\left(\frac{I_{or}}{2Rl} + \frac{I_{os}}{2Rd}\right) + \frac{1}{2}LR\pi\rho\Omega^2\left(\frac{A_s}{d} + \frac{A_r}{l}\right)$$
$$- \frac{L\pi A_r}{2lR}(E_r + N_\theta) - \frac{Ln^2\pi^2\lambda^2(G_r J_r + I_{or}N_\theta)}{2Rl} - \frac{1}{2d}LR\pi\lambda^4 I_{os}(E_s + N_a) - \frac{Ln^2\pi\lambda^2}{2Rd}(G_s J_s + I_{os}N_a)$$
$$- \frac{Ln^2\pi I_{or}}{2R^3 l}[N_\theta(1+n^2) + n^2 E_r] - \frac{1}{2d}LR\pi\lambda^2 A_s N_a - \frac{Ln^2\pi A_r N_\theta}{2Rl} - \frac{Ln^2\pi A_r z_r(E_r + 2N_\theta)}{R^2 l}$$
$$+ \left[L\pi\rho I_{os}\left(\frac{n^2}{2Rd} + \frac{R}{2d}\lambda^2\right) + L\pi\rho I_{or}\left(\frac{n^2}{2lR} + \frac{R\lambda^2}{2l}\right) + \frac{1}{2}LR\pi\rho\left(\frac{A_s}{d} + \frac{A_r}{l}\right)\right]\omega^2$$
$$- Ln\pi\rho\Omega\left(\frac{A_r z_r}{l} + \frac{A_s z_s}{d}\right)\omega$$

where $I_{os} = I_s + z_s^2 A_s$, $I_{or} = I_r + z_r^2 A_r$.

I_s and I_r are the moments of inertia of the stringers and rings about their centroidal axes, respectively, and the respective polar moments of inertia are

$$J_s = \tfrac{1}{3}b_s h_s^3, \qquad J_r = \tfrac{1}{3}b_r h_r^3$$

References

Bryan, G.H., 1890. On the beats in the vibration of a revolving cylinder or bell. Proceedings of the Cambridge Philosophical Society 7, 101–111.

DiTaranto, R.A., Lessen, M., 1964. Coriolis acceleration effect on the vibration of a rotating thin-walled circular cylinder. ASME Journal of Applied Mechanics 31, 700–701.

Egle, D.M., Sewall, J.L., 1968. An analysis of free vibration of orthogonally stiffened cylindrical shells with stiffeners treated as discrete elements. AIAA Journal 6, 518–526.

ESDU, 1982. Free vibrations of thin walled orthogonally stiffened circular cylindrical shells. ITEM No. 80040.

Huang, S.C., Chen, L.H., 1996. Vibration of a spinning cylindrical shell with internal/external ring stiffeners. ASME Journal of Vibration and Acoustics 118, 227–236.

Huang, S.C., Soedel, W., 1988. Effects of Coroilis acceleration on the forced vibration of rotating cylindrical shells. ASME Journal of Applied Mechanics 55, 231–233.

Lam, K.Y., Loy, C.T., 1995. Analysis of rotating laminated cylindrical shells by different thin shell theories. Journal of Sound and Vibration 186, 23–35.

Lee, Y.S., Kim, Y.W., 1998. Vibration analysis of rotating composite cylindrical shells with orthogonal stiffeners. Computers and Structures 69, 271–281.

Lee, Y.S., Kim, Y.W., 1999. Effect of boundary conditions on natural frequencies for rotating composite cylindrical shells with orthogonal stiffeners. Advanced Engineering Software 30, 649–655.

Mikulas Jr., M.M., McElman, J.A., 1965. On free vibrations of eccentrically stiffened cylindrical shells and flat plates. NASA TN D-3010.

Mustafa, B.A.J., Ali, R., 1989. An energy method for free vibration analysis of stiffened circular cylindrical shells. Computers and Structures 32, 355–363.

Parthan, S., Johns, D.J., 1970. Effects of inplane and rotary inertia on the frequencies of eccentrically stiffened cylindrical shells. AIAA Journal 8, 253–261.

Rinehart, S.A., Wang, J.T., 1972. Vibration of simply supported cylindrical shells with longitudinal stiffeners. Journal of Sound and Vibration 24, 151–163.

Rosen, A., Singer, J., 1974. Vibrations of axially loaded stiffened cylindrical shells. Journal of Sound and Vibration 34, 357–378.

Srinivasan, A.V., Lauterbach, G.F., 1971. Travelling waves in rotating cylindrical shells. Journal of Engineering for Industry 93, 1229–1232.